Indian Statistical Institute
Mid-Semestral Examination
Topology II - MMath I

Max Marks: 40 Time: 180 minutes.

Answer all questions. In what follows "E, X" will always denote spaces that are path connected and locally path connected.

(1) Decide whether the following statements are TRUE or FALSE. Justify. Answers not accompanied by a correct justification will not be awarded any marks.
  (a) If $p : E \longrightarrow X$ is a covering and $p_* : \pi_1(E, e) \longrightarrow \pi_1(X, p(e))$ is surjective, then p is a homeomorphism.
  (b) Let X be the space obtained from S^2 by identifying $(0, 0, 1)$ with $(0, 1, 0)$. Then X is simply connected.
  (c) There exists a covering $p : E \longrightarrow S^1 \vee S^1$ with $\mathrm{Cov}(E/S^1 \vee S^1) \cong \mathbb{S}_3$, the symmetric group on three letters.
  (d) If X has a universal cover, then X is semi-locally simply connected. $[4 \times 2]$

(2) Let Y denote the following subspace of $\mathbb{R}^2$:

$$Y = (\{0\} \times [0, 1]) \cup (\{1/n\}_{n \geq 1} \times [0, 1]) \cup ([0, 1] \times \{0\}).$$

Show that there exists a homotopy $F_t : Y \longrightarrow Y$ with $F_0 = 1_X$ (the identity map of X) and $F_1 = c_{(0,1)}$ (the constant map at $(0, 1)$). Show that any such homotopy cannot in addition satisfy $F_t(0, 1) = (0, 1)$ for all $t \in [0, 1]$. $[4 + 4]$

(3) Show that the following statements are equivalent for a map $f : S^n \longrightarrow X$.
  (a) f is homotopic to constant.
  (b) f can be extended to a map $D^{n+1} \longrightarrow X$.
  (c) If $x_0 \in S^n$ and $c : S^n \longrightarrow X$ is the constant map at $f(x_0)$, then there is a homotopy $F : f \sim c$ with $F(x_0, t) = f(x_0)$ for all $t \in [0, 1]$.

Recall that D^{n+1} denotes the set of elements $x \in \mathbb{R}^{n+1}$ with $||x|| \leq 1$. [8]

(4) Define the term : *regular covering*. Show that a covering $p : E \longrightarrow X$ is regular if and only if for each closed path $\alpha : [0, 1] \longrightarrow X$, either every lifting $\widetilde{\alpha}$ of α is a closed path or no lifting $\widetilde{\alpha}$ of α is a closed path. [2+6]

(5) Let $F_2 = \mathbb{Z} * \mathbb{Z}$ be the free product of two copies of $\mathbb{Z}$, the first copy generated by a and the second generated by b. Let $H = \langle\langle\{ab, ba, a^4\}\rangle\rangle$ be the normal closure of $\{ab, ba, a^4\}$ in F_2. Then show that H has finite index in F_2. [8]